ALGEBRAS WITHOUT NOETHERIAN FILTRATIONS

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Abstract. We provide examples of finitely generated noetherian PI algebras for which there is no finite dimensional filtration with a noetherian associated graded ring; thus we answer negatively a question of Lorenz [6, p.436].

1. Introduction

In this paper all algebras will be defined over a fixed base field k. Let Γ be an $\mathbb{N}$-filtration of a k-algebra A; thus, $\Gamma = \{\Gamma_i : i \in \mathbb{N}\}$ is an ascending chain of k-subspaces of A satisfying $1 \in \Gamma_0$, $\bigcup_{j\in\mathbb{N}} \Gamma_j = A$ and $\Gamma_i\Gamma_j \subseteq \Gamma_{i+j}$ for all i, j. The filtration is *finite* if $\dim_k \Gamma_i < \infty$ for all $i \in \mathbb{N}$ and *standard* if $\Gamma_0 = k$ and $\Gamma_i = \Gamma_1^i$ for all $i \geq 2$. We say that Γ is a *(left) noetherian filtration* if the associated graded ring $\operatorname{gr} A = \operatorname{gr}_\Gamma A = \bigoplus_i \Gamma_i/\Gamma_{i-1}$ is (left) noetherian. The algebra A is *affine* if it is finitely generated as a k-algebra.

If an algebra A has a left noetherian $\mathbb{N}$-filtration, then a standard technique is to pull results back from $\operatorname{gr} A$ to A since theorems are typically easier to prove in the graded ring $\operatorname{gr} A$ than in A. For related reasons Lorenz asked in [6, p.436] and [7, Question III.4.2] whether every left noetherian affine algebra R, satisfying a polynomial identity (PI), admits a left noetherian, standard finite $\mathbb{N}$-filtration. This question has been raised again (without the "standard" hypothesis) in [15, Question 6.16] because of its importance for dualizing complexes and homological questions: if Lorenz's question were to have a positive answer then R would have a dualizing complex [15, Corollary 6.9] and every noetherian affine PI Hopf algebra would have finite injective dimension [13].

The aim of this note is to answer these questions by providing a class of noetherian affine PI algebras which do not admit any noetherian finite $\mathbb{N}$-filtration. The basic technique is provided by the following theorem (see Section 2).

Theorem 1.1. *Suppose that I and $J_1 \subset J_2$ are ideals of an algebra R such that J_2/J_1 is free of rank s as a left R/I-module and free of rank t as a right R/I-module. If $s < t$, then there is no finite $\mathbb{N}$-filtration Γ of R such that $\operatorname{gr}_\Gamma R$ is left noetherian.*

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A variant of this theorem also holds if one replaces "free of rank x" by "of Goldie rank x." See Theorem 3.2 for the details.

A simple example satisfying the hypotheses (and conclusion) of the theorem is given by the ring

$$R = \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x^2) \end{pmatrix} : f, g \in k[x] \right\} \subset M_2(k[x]). \tag{1.2}$$

Here, one takes $J_2 = I$ to be the ideal of strictly upper triangular matrices and $J_1 = 0$. See Section 4 for more details.

The most important case of Theorem 1.1 is when Γ is a standard. However, we emphasize that the theorem holds for any filtration satisfying the earlier definitions.

The analogue of Theorem 1.1 also holds for $\mathfrak{m}$-adic filtrations (where $\mathfrak{m}$ is the Jacobson radical of a local algebra) and for that reason we prove the result for rings with a Zariskian filtration (see Theorem 2.4). In particular, we provide an example of a local prime noetherian PI ring for which the Jacobson radical does not satisfy the strong AR property. This is given in Section 4 where the reader may find further applications of the main theorem.

2. Proof of Theorem 1.1

Since we are also interested in $\mathfrak{m}$-adic filtrations of local rings as well as ascending filtrations, we will prove our main result for $\mathbb{Z}$-filtrations. First we review some basic facts about filtrations from [3, Chapter 6] and [4].

Suppose that A is a k-algebra. For the purposes of this paper a *filtration* (or more strictly, an exhaustive separated finite $\mathbb{Z}$-filtration) of A is an ascending chain of subspaces $\Gamma = \{\Gamma_i \subseteq \Gamma_{i+1} \mid i \in \mathbb{Z}\}$ of A, satisfying:

(1) $1 \in \Gamma_0$ and $\Gamma_i \Gamma_j \subseteq \Gamma_{i+j}$ for all $i, j \in \mathbb{Z}$;
(2) Γ is *finite* in the sense that $\dim \Gamma_i/\Gamma_{i-1} < \infty$ for all $i \in \mathbb{Z}$;
(3) Γ is *exhaustive* in the sense that $A = \bigcup_{i\in\mathbb{Z}} \Gamma_i$ and *separated* in the sense that $\bigcap_{i\in\mathbb{Z}} \Gamma_i = 0$.

The *Rees ring* associated to Γ is defined to be $\operatorname{Rees} A = \operatorname{Rees}_\Gamma A = \bigoplus_{i\in\mathbb{Z}} \Gamma_i$ and the *associated graded ring* is $\operatorname{gr} A = \operatorname{gr}_\Gamma A = \bigoplus_{i\in\mathbb{Z}} \Gamma_i/\Gamma_{i-1}$. Write $J(A)$ for the Jacobson radical of A. Following [4], a filtered algebra A is called *(left) Zariskian* if

(Zar1) $\Gamma_{-1} \subseteq J(\Gamma_0)$;
(Zar2) $\operatorname{Rees}_F A$ is (left) noetherian.

Fix a filtration Γ of the algebra A and a left A-module M. The concept of a $\mathbb{Z}$-filtration (again, finite, exhaustive and separated) $\Lambda = \{\Lambda_i : i \in \mathbb{Z}\}$ of M is defined analogously; one simply replaces (1) in the above definition by

(1′) $\Gamma_i \Lambda_j \subseteq \Lambda_{i+j}$ for all $i, j \in \mathbb{Z}$.

Corresponding to this filtration one has the Rees module $\mathrm{Rees}_\Lambda M = \bigoplus \Lambda_i$ and associated graded module $\mathrm{gr}_\Lambda M = \bigoplus \Lambda_i/\Lambda_{i-1}$. We say that Λ is a *good filtration* if there exist $\{m_i \in \Lambda_{d_i} : 1 \le i \le r < \infty\}$ such that $\Lambda_n = \sum_{i=1}^r \Gamma_{n-d_i} m_i$ for all $n \in \mathbb{Z}$. Two filtrations Λ and Λ' of M are *equivalent*, written $\Lambda \sim \Lambda'$, if there is an integer q such that $\Lambda_i \subset \Lambda'_{i+q}$ and $\Lambda'_i \subset \Lambda_{i+q}$ for all $i \in \mathbb{Z}$.

The *Hilbert function* of M with respect to Λ is defined to be

$$H_{M,\Lambda}(n) = \dim \Lambda_n/\Lambda_{-n}, \quad \text{for all } n \in \mathbb{Z}.$$

Let H and H' be two (Hilbert) functions $\mathbb{N} \to \mathbb{N}$. We say that the growth of H is at most the growth of H', and write $H \le H'$, if there is an integer q such that $H(n) \le H'(n+q)$ for all $n \gg 0$. We say H and H' are *equivalent*, written $H \sim H'$, if both $H \le H'$ and $H' \le H$ hold.

Since we require filtrations to be separated, they need not induce filtrations on factor modules. However, for Zariskian filtrations this is not a problem:

Lemma 2.1. *Suppose that $\Gamma = \{\Gamma_i\}$ is a filtration of an algebra A and let M be a left A-module with a good filtration Λ.*

(1) *If Λ' is another filtration of M, there exists an integer q such that $\Lambda_i \subset \Lambda'_{i+q}$ for all i. Thus, if Λ' is good, then Λ and Λ' are equivalent.*
(2) *If Λ' and Λ'' are equivalent filtrations of M, then $H_{M,\Lambda'}$ and $H_{M,\Lambda''}$ are equivalent.*
(3) *Assume that Γ is left Zariskian. If N is a submodule of M then Λ induces good filtrations on N and M/N. In particular, Γ induces a left Zariskian filtration on every factor ring of A.*

Proof. (1) and (2) follow from the definitions while (3) follows from [4, Theorem 3.3] or [5, Theorem II.2.1.2]. □

The key observation in this paper is given by the next proposition. Since our main theorem will come in three slightly different forms, this result will also have three slightly different cases.

Proposition 2.2. *Let A be an algebra with a filtration Γ such that the growth of $H_{A,\Gamma}$ is subexponential. Let M be an A-bimodule such that the left module ${}_AM$ is free of rank s and the right module M_A is free of rank t. Suppose that there exists a filtration Λ on M such that Λ is a good filtration of ${}_AM$ and a filtration of M_A.*

(1) *If Λ is also a good filtration of M_A then $t = s$.*
(2) *If $\Gamma_n = 0$ for $n \ll 0$ then $s \ge t$*
(3) *If $\Gamma_n = A$ for $n \gg 0$ then $s \le t$.*

Proof. The beginning of the proof is the same in all three cases. Note that ${}_AA$ has a good filtration, simply because $\Gamma_i = \Gamma_i 1$ for all i. Thus ${}_AA^{(s)}$ also has a good filtration and so this induces a good filtration Λ' on M such that $\Lambda'_i \cong \Gamma_i^{(s)}$ for all i. By Lemma 2.1(1), there exists q_1 such that $\Lambda_i \subseteq \Lambda'_{i+q_1}$ for all i.

Similarly, the right A-module structure provides an induced good filtration Λ'' of M_A such that $\Lambda''_i \cong \Gamma_i^{(t)}$ for all i. By Lemma 2.1(1), there exists q_2 such that $\Lambda''_i \subseteq \Lambda_{i+q_2}$ for all i. Hence, for $p = q_1 + q_2$,

$$\Lambda''_i \subseteq \Lambda'_{i+p} \quad \text{for all } i \in \mathbb{Z}. \tag{2.3}$$

We now consider the three cases separately. Under assumption (2), we see that $\Lambda''_i/\Lambda''_{-i} = \Lambda''_i \hookrightarrow \Lambda'_{i+p} = \Lambda'_{i+p}/\Lambda'_{-(i+p)}$ for $i \gg 0$. Thus, $H_{M,\Lambda''}(n) \leq H_{M,\Lambda'}(n+p)$. But, by construction, $H_{M,\Lambda'} = sH_{A,\Gamma}$ and $H_{M,\Lambda''} = tH_{A,\Gamma}$ and so

$$H_{A,\Gamma}(n) \leq \frac{s}{t} H_{A,\Gamma}(n+p) \quad \text{for all } n \in \mathbb{N}.$$

Since $H_{A,\Gamma}(n)$ grows subexponentially this forces $t \leq s$.

Under assumption (3) we have $\Lambda''_{i+p}/\Lambda''_{-(i+p)} = A/\Lambda''_{-i-p} \twoheadrightarrow A/\Lambda'_{-i} = \Lambda'_i/\Lambda'_{-i}$ for $i \gg 0$. Thus, $H_{M,\Lambda'}(i) \leq H_{M,\Lambda''}(i+p)$ and repeating the analysis of the last paragraph shows that $s \leq t$.

Finally, assume that (1) holds. In this case, $\Lambda'' \sim \Lambda$ and so $\Lambda'' \sim \Lambda'$. Thus there exists p such that $\Lambda'_i \subseteq \Lambda''_{i+p} \subseteq \Lambda'_{i+2p}$. Now a minor variant of the penultimate paragraph shows that $t \leq s$ and hence, by symmetry, that $s = t$. □

Theorem 2.4. *Suppose that R is an algebra with ideals I and $J_1 \subset J_2$ such that J_2/J_1 is free of rank s as a left R/I-module and free of rank t as a right R/I-module. If $s < t$, then there is no filtration of R such that R is both left and right Zariskian.*

Proof. Let $\widetilde{\Gamma}$ be a left and right Zariskian filtration of R. Then Lemma 2.1(3) implies that the induced filtration Γ on $A = R/I$ is left and right Zariskian and so $\operatorname{gr}_\Gamma A$ is left (and right) noetherian. By [12, Remark after 1.2] the growth of $H_{A,\Gamma}$ is subexponential. By Lemma 2.1(3), Γ induces a good filtration on J_2/J_1 as a left and a right A-module, which contradicts to Proposition 2.2(1). □

A curious feature of this result is that R could be left or right Zariskian; it just cannot be both (see Corollary 4.7). However, with a little more information one can determine which side goes wrong.

Theorem 2.5. *Suppose that R is an algebra with ideals I and $J_1 \subset J_2$ such that J_2/J_1 is free of rank s as a left R/I-module and free of rank t as a right R/I-module. Assume that $s < t$ and that Γ is a filtration of R.*

(1) *If $\Gamma_i = 0$ for $i \ll 0$, then Γ is not left Zariskian.*
(2) *If $\Gamma_i = A$ for $i \gg 0$, then Γ is not right Zariskian.*

Proof. For (1), repeat the proof of Theorem 2.4, but with Proposition 2.2(1) replaced by Proposition 2.2(2). For (2), use the proof of Theorem 2.4, applied to the opposite ring R^{op}, with Proposition 2.2(1) replaced by Proposition 2.2(3). □

We are now ready to prove Theorem 1.1 from the introduction.

Proof of Theorem 1.1. It is easy to see that an $\mathbb{N}$-filtration Γ of a ring R is left Zariskian if and only if gr_Γ is left noetherian (see, for example, [5, Proposition II.1.2.3]). Thus, Theorem 1.1 is a special case of Theorem 2.5(1). $\square$

The analogue of Theorem 1.1 for complete local rings also holds with much the same proof. To state the result, we need some definitions. Let R be a semilocal algebra with Jacobson radical $J(R) = \mathfrak{m}$ and assume that $\dim_k R/\mathfrak{m} < \infty$. A filtration Γ of R is called a *weak adic filtration* if it satisfies $\Gamma_n = A$ for all $n \geq 0$ and $\Gamma_{-1} \subseteq \mathfrak{m}$. (We still require the filtration to be finite, separated and exhaustive.)

Corollary 2.6. *Let R be a complete semilocal right noetherian algebra with Jacobson radical $\mathfrak{m}$. Suppose that R has ideals I and $J_1 \subset J_2$ such that J_2/J_1 is free of rank s as a left R/I-module and free of rank t as a right R/I-module. If $s < t$, then there is no weak adic filtration Γ such that $\mathrm{gr}_\Gamma R$ is right noetherian.*

Proof. Suppose that such a filtration Γ exists. By [2, Corollary 1.2] the filtration Γ is also complete (in the natural sense that Cauchy sequences modulo the Γ_{-i} should converge—see [5, Definition I.3.3.2]). Thus [5, Proposition II.2.2.1] implies that the filtration is right Zariskian and the result follows Theorem 2.5(2). $\square$

An analogue of this corollary also holds for non-complete rings, although the result is less pleasant since one cannot now assume that separated filtrations induce separated filtrations on factor modules. The result becomes the following: Suppose that R satisfies the hypotheses of Corollary 2.6, but that R is not complete. Let Γ be a weak adic filtration of R that induces both a filtration on R/I and a good right filtration of J_2/J_1. If $s < t$, then $\mathrm{gr}_\Gamma R$ is not right noetherian.

3. A partial generalization

Although the results of the last section are sufficient for our examples, one can give a version of Theorem 1.1 that works without the assumption that J_2/J_1 be free, but at the expense of of assuming that R/I be a prime Goldie ring. We prove this in this section. Let A be a prime left Goldie ring with simple artinian ring of fractions $Q(A)$. If M is a left A-module, then the *Goldie rank* of M is defined to be the length of $Q(A) \otimes_A M$ and written $\mathrm{Grank}(M)$.

Lemma 3.1. *Suppose that A is a prime Goldie ring with a left noetherian $\mathbb{N}$-filtration Γ. Let M be an A-bimodule with a filtration Λ that is a good filtration of ${}_AM$ and a filtration of M_A. If ${}_AM$ is torsion, then so is M_A.*

Proof. Assume that M_A is not torsion. Replacing M by $M^{(n)}$, for some n, we may assume that $A_A \subseteq M_A$. Thus, Lemma 2.1(1) implies that there exists $q \geq 0$ such that

$$\dim \Gamma_n \leq \dim \Lambda_{n+q} \cap A \leq \dim \Lambda_{n+q}$$

for all n. Since $\Gamma_n = \Lambda_n = 0$ for $n \ll 0$, this implies that $H_{M,\Lambda} \geq H_{A,\Gamma}$.

Now consider ${}_AM$. Since Λ is a good filtration, ${}_AM$ is finitely generated. We claim, for all $p > 0$, that $H_{M,\Lambda} \leq \frac{1}{p} H_{A,\Gamma}$. Once this has been proved, then the last paragraph implies, for some $x > 0$, that $H_{A,\Gamma}(n) \leq \frac{1}{p} H_{A,\Gamma}(n+x)$. This contradicts the fact that $H_{A,\Gamma}$ grows subexponentially [12, Remark after 1.2] and proves the lemma. In order to prove the claim we ignore the right-hand structure of M and so, by induction, it suffices to prove it for a cyclic module $M = A/I$. Since ${}_AM$ is torsion and A is Goldie, I contains a regular element, a say, of A. We will still write Γ for the good filtration on any subfactor of ${}_AA$ induced from Γ. Now, for any n, M is a homomorphic image of $L(i) = Aa^{i-1}/Aa^i \cong A/Aa$ and so $H_{M,\Lambda} \leq H_{L(i),\Gamma}$. Since Hilbert series are additive on short exact sequences, this implies that $pH_{M,\Lambda} \leq H_{A/Aa^p,\Gamma} \leq H_{A,\Gamma}$, for any $p \in \mathbb{N}$. $\square$

Theorem 3.2. *Suppose that I and $J_1 \subset J_2$ are ideals of an algebra R such that $A = R/I$ is a prime Goldie ring. Assume that J_2/J_1 has Goldie rank s as a left A-module and Goldie rank t as a right A-module, for some $s < t$. Then there is no $\mathbb{N}$-filtration Γ' of R such that $\mathrm{gr}_{\Gamma'} R$ is left noetherian.*

Proof. Suppose that such a filtration Γ' exists. Let Γ be the induced filtration on A and Λ the induced filtration on $M = J_2/J_1$. Then Λ is a good left filtration and so ${}_AM$ is finitely generated. The torsion submodule T of M_A is an A-bimodule and so we may pass to M/T without affecting the hypotheses (although s may decrease). By Lemma 3.1 ${}_AM$ is also torsion-free. If $\mathrm{Grank}(A) = u$, replace M by $M^{(u)}$; thus $A_A^{(t)} \cong X \subseteq M_A$ and ${}_AM \subseteq Y \cong {}_AA^{(s)}$. Let Θ, respectively Φ, be the filtrations of X and Y induced from Γ. Since Θ equals the direct sum $\Gamma^{(t)}$ of t copies of Γ, certainly $H_{X,\Theta} = tH_{A,\Gamma}$. Similarly, $H_{Y,\Phi} = sH_{A,\Gamma}$.

Since Λ is a good left filtration and Θ is a good right filtration, by Lemma 2.1(1) there exists $q \geq 0$ such that

$$\dim \Theta_n \leq \dim(X \cap \Lambda_{n+q}) \leq \dim \Lambda_{n+q}$$

and

$$\dim \Lambda_n \leq \dim(M \cap \Phi_{n+q}) \leq \dim \Phi_{n+q},$$

for all $n \gg 0$. Hence,

$$tH_{A,\Gamma}(n) = H_{X,\Theta}(n) \leq H_{Y,\Phi}(n+2q) = sH_{A,\Gamma}(n+2q),$$

for all $n \gg 0$. Since $H_{A,\Gamma}$ grows subexponentially, this forces $t \leq s$. $\square$

With a rather more complicated argument one can prove an analogous version of Theorem 2.4 using Goldie ranks. However, we do not know how to prove the analogous version of Theorem 2.5(2) or Corollary 2.6 without extra hypotheses.

4. Examples

In this section we use the results of the Section 2 to provide examples of rings without noetherian associated graded rings.

Example 4.1. *Let*

$$S = \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x^2) \end{pmatrix} : f, g \in k[x] \right\} + yM_2(k[x,y]) \ \subset \ M_2(k[x,y]).$$

Then S is an affine noetherian prime PI algebra without any left noetherian finite $\mathbb{N}$-filtrations.

Proof. The diagonal matrices of the form

$$\begin{pmatrix} f(x,y) & 0 \\ 0 & f(x^2,y) \end{pmatrix} : f(x,y) \in k[x,y]$$

clearly form a subring C of S isomorphic to $k[x,y]$. Since S is finitely generated as a left or right C-module, it follows that S is a finitely generated noetherian PI algebra. It is prime since it contains a nonzero ideal of the prime ring $M_2(k[x,y])$.

Suppose that $A \to B$ is a surjective homomorphism of algebras and that B does not admit a left noetherian $\mathbb{N}$-filtration. Then an immediate consequence of Lemma 2.1(3) is that A also has no such filtration. Thus, it suffices to prove the final assertion for a factor ring, and we use

$$R = S/yM_2(k[x,y]) \ \cong \ \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x^2) \end{pmatrix} : f, g \in k[x] \right\} \ \subset \ M_2(k[x]). \tag{4.2}$$

Notice that this is the ring from (1.2). The nilradical $N = N(R)$ is just the set of strictly upper triangular matrices. It is routine to check that N is a free left R/N-module of rank one but a free right R/N-module of rank two. Hence, by Theorem 1.1, neither R nor S can have a left noetherian finite $\mathbb{N}$-filtrations. □

Remark 4.3. The ring R from (4.2) clearly has Gelfand-Kirillov dimension one. By inspecting the proof of Theorem 2.4, this shows that there is no finite filtration of R that induces a good filtration of $N(R)$ as a left R-module.

Other examples of noetherian rings with no noetherian $\mathbb{N}$-filtration are given in [12]. However, those examples require that the ring in question have infinite Gelfand-Kirillov dimension and this is impossible for affine PI algebras (see [8, Proposition 13.10.6]).

It is readily checked that this ring R does have a right noetherian, finite filtration (see Corollary 4.7). By slightly modifying the example one can get an example that "works" on both sides.

Example 4.4. *Let*

$$T = \left\{ \begin{pmatrix} f(x) & g(x) & h(x) \\ 0 & f(x^2) & l(x) \\ 0 & 0 & f(x) \end{pmatrix} \; : \; f(x), \dots, l(x) \in k[x] \right\} + yM_3(k[x,y]).$$

Then T is a affine noetherian prime PI algebra such that, for every $\mathbb{N}$-filtration, the associated graded ring is neither left nor right noetherian.

Proof. Notice that $T/(e_{13}T + e_{23}T) \cong R$, the ring from (4.2), and so T has no left noetherian finite filtration. Since $T \cong T^{op}$, the same is true on the right. □

These examples can be easily modified into complete local rings and we give one example that is analogous to the ring R from Example 4.1. Before giving the example, we need a definition. An ideal I of a ring R is said to satisfy the *strong AR* property if the associated Rees ring $\mathrm{Rees}_I\, R = \bigoplus_{j\geq 0} I^j$ is noetherian. The significance of this condition is that, if I satisfies the strong AR property, then it also satisfies the usual Artin-Rees (AR) property; indeed this is the standard way of proving the latter condition in commutative algebra. The same idea has been useful in noncommutative algebra (see, for example, [10, §2] and [5]). Given that the Jacobson radical of a semi-local noetherian PI algebra is automatically AR ([1, Theorems 3.1.13 and 7.2.5]), it is natural to ask if it is also strongly AR.

The next example shows that it does not. Notice that, although the ring in question is just a completion of the ring R from Example 4.1, the Zariskian property fails on the opposite side.

Example 4.5. *Let*

$$R = \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x^2) \end{pmatrix} : f, g \in k[[x]] \right\} + yM_2(k[[x,y]]) \;\subset\; M_2(k[[x,y]]).$$

Then R a prime noetherian complete local PI algebra over k. Moreover, R has no weak adic filtration Γ such that $\mathrm{gr}_\Gamma\, R$ is right noetherian. In particular, $J(R)$ does not satisfy the strong AR property.

Proof. The proof that R is prime noetherian PI algebra over k is analogous to that for Example 4.1 and is left to the reader. Clearly R is a complete local ring.

In order to complete the proof it suffices, by Corollary 2.6, to work in a factor ring and we chose:

$$R' = R/yM_2(k[[x,y]]) \;=\; \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x^2) \end{pmatrix} : f(x), g(x) \in k[[x]] \right\}. \tag{4.6}$$

The nilradical N of R' is the set of strictly upper triangular matrices and is free of rank 1 as a left R'/N-module and free of rank 2 as a right R'/N-module. Now apply Corollary 2.6. □

Finally we justify a remark made after Theorem 2.4: in that theorem it is possible to have a filtration that is Zariskian on either the left or the right.

Corollary 4.7. *Consider noetherian, PI algebras R with ideals I and $J_1 \subseteq J_2$ such that J_2/J_1 is a free left R/I-module of rank one and a free right R/I-module of rank* 2.

Then, there exists an example R_1 of such a ring with a left Zariskian filtration Γ_1 and an example R_2 of such a ring with a right Zariskian filtration Γ_2.

Proof. The ring R_2 is just the ring R from (1.2), and so does satisfy the hypotheses of the first paragraph. Set

$$\alpha = \begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.8}$$

It is easy to see that R is generated by α and β and hence one has the standard $\mathbb{N}$-filtration $\Gamma_0 = k$, $\Gamma_1 = k + k\alpha + k\beta$ and $\Gamma_n = \Gamma_1^n$ for $n \geq 2$. Let a and b be the images of α and β in $\mathrm{gr}_\Gamma R$. Then, $\mathrm{gr}_\Gamma R$ is generated by a and b and satisfies the relations $a^2b = 0 = b\,\mathrm{gr}_\Gamma Rb$. Thus $\mathrm{gr}_\Gamma R$ is spanned by the elements $\{a^n, ba^n, aba^n \ : n \geq 0\}$. As such, $\mathrm{gr}_\Gamma R$ is a finitely generated right $k[a]$-module, and so is right noetherian. As in the proof of Theorem 1.1, this suffices to prove that Γ is right Zariskian.

For R_1 we take the ring R' from (4.6). We define α and β by (4.8) and observe that $\mathfrak{m} = J(R') = R'\alpha + R'\beta$. Now use the $\mathfrak{m}$-adic filtration Γ defined by $\Gamma_i = R'$ if $i \geq 0$ but $\Gamma_i = \mathfrak{m}^{-i}$ if $i \leq 0$. Then $\beta\alpha = \alpha^2\beta \in \mathfrak{m}^3$. Hence in the associated graded ring one finds that the images of these elements satisfy $ba = 0 = b^2$. The argument of the last paragraph shows that $\mathrm{gr}_\Gamma R'$ is left noetherian and hence, by [5, Proposition II.1.2.3], that Γ is left Zariskian. □

5. A Dualizing Module

As was remarked in the introduction, if an affine noetherian PI algebra R has a finite noetherian $\mathbb{N}$-filtration, then R has a dualizing complex. Thus, one can ask whether the ideas of the last section can be used to provide examples of PI rings which do not have such a complex. This appears not to be the case; in this section we check that the ring R from (1.2) does indeed have such a complex.

One advantage of this example is that we can work with modules rather than complexes, and so we define an (R, R)-bimodule D to be a *dualizing module* if (i) D is finitely generated and of finite injective dimension on both sides and (ii) the natural maps $R \to \mathrm{End}(D_R)$ and $R^{\mathrm{op}} \to \mathrm{End}({}_RD)$ are isomorphisms. A dualizing module viewed as a complex is a dualizing complex in the sense of Yekutieli [14].

The way we find a dualizing module for the ring R from (1.2) is through the following observation: Identify $C = k[x]$ with the diagonal matrices in R and set

$$D_1 = \mathrm{Hom}_C({}_CR, C) \qquad \text{and} \qquad D_2 = \mathrm{Hom}_C(R_C, C). \tag{5.1}$$

Thus, D_1 is an (R, C)-bimodule and D_2 is a (C, R)-bimodule. The key to our construction is the following easy lemma.

Lemma 5.2. *Let $R \supseteq C$ be rings such that R is a finitely generated projective C-module on both sides and C is a commutative noetherian algebra of finite injective dimension. Define modules D_i by* (5.1)*. Suppose that one has ring isomorphisms $\mathrm{End}_R(D_1) \cong R^{\mathrm{op}}$ and $\mathrm{End}_R(D_2) \cong R$ through which $D_1 \cong D_2$ as R-bimodules. Then, $D = D_1$ is a dualizing module for R.*

Proof. It is clear that C is a dualizing module for itself. By [9, Theorem 11.66] one has natural isomorphisms:

$$\mathrm{Ext}^i_R({}_RN, {}_RD_1) = \mathrm{Ext}^i_R({}_RN, \mathrm{Hom}_C({}_CR, C)) \cong \mathrm{Ext}^i_C({}_CN, C), \tag{5.3}$$

for any finitely generated left R-module N. This implies that the injective dimension of ${}_RD_1$ is bounded by the injective dimension of ${}_CC$. A similar assertion holds for $(D_2)_R$. It follows from [11, Theorem 3.5] that ${}_RD_1$ and $(D_2)_R$ are finitely generated. Therefore $D = D_1$ is a dualizing module. □

Proposition 5.4. *Let R be the ring defined by* (1.2) *and define modules D_i by* (5.1)*. Then, $D = D_1$ is a dualizing module for R.*

Proof. We check that the hypotheses of the lemma are satisfied for $C = k[x]$. Since R is a free left $k[x]$-module of rank two and a free right $k[x]$-module of rank three, the proof is a routine computation which we will only outline. One first checks that, under the natural module structures,

$$D_1 \cong \frac{R \oplus R}{R(xe_{12}, e_{12})} \qquad \text{and} \qquad D_2 \cong R/e_{12}R,$$

where $e_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the matrix unit. It follows that

$$\mathrm{End}_R(D_2) \cong \mathbb{I}(e_{12}R)/e_{12}R \qquad \text{where} \quad \mathbb{I}(aR) = \{\theta \in R : \theta aR \subseteq aR\}.$$

Computing this out one finds that

$$\mathrm{End}_R(D_2) \cong \left\{ \begin{pmatrix} f(x^2) & g(x) \\ 0 & f(x^4) \end{pmatrix} : f, g \in k[x] \right\} \Big/ e_{12}k[x^2]$$

$$\cong \left\{ \begin{pmatrix} f(x^2) & g(x^2) \\ 0 & f(x^4) \end{pmatrix} : f, g \in k[x] \right\} \cong R.$$

Finally, under this isomorphism $E = \mathrm{End}_R(D_2) \cong R$ one finds that the modules ${}_ED_2$ and ${}_RD_1$ become isomorphic, as is required to prove the proposition. □

It would be interesting to know whether this proof can be extended to work for any ring S that is finitely generated as a module over a commutative subring. The fact that the present proof depends upon the "lucky" isomorphism $\mathrm{End}_R(D_2) \cong R$

makes this seem unlikely. By using ideas from [13], it can at least be extended to Hopf algebras finitely generated as modules over commutative subalgebras.

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